9 How to study $D(\mathrm{Bun}_G)$

<u>Last time</u> $G = \mathbb{G}_m = GL_1$

$$\begin{array}{ccccccc} D(\mathrm{Bun}_{\mathbb{G}_m}) & \simeq & D(\mathrm{Jac}) & \otimes & D(\mathbb{Z}) & \otimes & D(B\mathbb{G}_m) \\ \wr\!\!\! \mid & & \wr\!\!\! \mid & & \wr\!\!\! \mid & & \wr\!\!\! \mid \\ QC(\mathrm{Flat}_{\mathbb{G}_m}) & \simeq & QC(\underline{\mathrm{Flat}_1}) & \otimes & QC(\mathbb{G}_m) & \otimes & QC \end{array}$$

For $G = T$ torus

$$\mathrm{Bun}_T = \coprod_{X} \mathrm{Jac}_T \times B\mathbb{G}_m$$

$X \leftarrow$ cocharacter lattice $= \{\mathbb{G}_m \to T \mid \text{homo}\}$

<u>Fact</u> $\pi_0(\mathrm{Bun}_G) \simeq \pi_0(\mathrm{Gr}_G) = \pi_1(G) = \Lambda_G / \check{R}_G$

$\Lambda_G \uparrow$ coweights, $\check{R}_G \uparrow$ coroots

$\Rightarrow \pi_0(\mathrm{Bun}\, T) = \pi_1(T) = \Lambda_T$

How about Bun_G?

G is combinatorially complicated object and there is no such easy description

Idea: use easier groups attached to G

$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \leftrightarrow B$ Borel subgroup (solvable)

$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \leftrightarrow N$ unipotent radical

1) Generic Reductions

$\mathrm{Bun}_G(C) : (\mathrm{Sch}^{\mathrm{aff}})^{op} \to \mathrm{Spc}$

$S \to \{\underbrace{P_G}_{\text{Principle } G\text{-bundle}} \text{ on } C \times S = G\}$

scheme $X \rightsquigarrow \mathrm{Hom}(-, X)$

$\mathrm{Bun}_G C \rightsquigarrow \mathrm{Hom}(-, \mathrm{Bun}_G C)$

$= \mathrm{Hom}(-, \mathrm{Hom}(C, BG))$

$= \mathrm{Hom}(C \times (-), BG)$

definition of Bun_G in Alg. Geom note $\mathbb{B}G : (\mathrm{Sch}^{\mathrm{aft}})^{op} \to \mathrm{Spc}$

$$S \to \mathrm{Bun}_G(S)_{spc} \leftarrow \text{no idea}$$
$$C \to \mathrm{Bun}_G(C)_{spc} \leftarrow \text{about algebraic structure}$$

definition of Bun_G in Alg. Top

$\mathrm{Bun}_B : S \mapsto \{P_B \text{ on } C_S\}$

Plücker description of Bun_B

B-bundle + flag section is flag of bundles

$B \hookrightarrow G$ ~~$\mathbb{B}B$~~

$$\begin{tikzcd}
G/B \arrow[r] \arrow[d, "\Gamma"'] & \mathbb{B}B = pt/B \arrow[d] \\
pt \arrow[r] & \mathbb{B}G = pt/G \\
C \arrow[ur, bend left] &
\end{tikzcd}$$

$G = GL_n$ $\quad V =$ defining rep

$W \subset V$ dim k

$\Lambda^k W \subset \Lambda^k V$ or $\Lambda^k W \in \mathbb{P}(\Lambda^k V)$

line

$Gr(k,n) \hookrightarrow \mathbb{P}(\Lambda^k V)$

Plücker embedding

$W_1 \subset \cdots \subset W_n = V$

$$Fl(V) \hookrightarrow \prod_{k=1}^{n} \mathbb{P}(\Lambda^k(V))$$

$$\begin{tikzcd}
\Lambda^i V \otimes \Lambda^j V \arrow[rr, "f_{ij}"] \arrow[dr] & \circlearrowright & \Lambda^{i-1} V \otimes \Lambda^{j+1} V \\
& \Lambda^{i-1} V \otimes V \otimes \Lambda^j V \arrow[ur] &
\end{tikzcd}$$

$f_{11}(v_1 \otimes v_2) = v_1 \wedge v_2$

$f_{12}(v \otimes (v_1 \wedge v_2)) = v \wedge v_1 \wedge v_2$

$f_{22}((v_1 \wedge v_2) \otimes (v_3 \wedge v_4)) = v_1 \otimes (v_2 \wedge v_3 \wedge v_4) - v_2 \otimes (v_1 \wedge v_3 \wedge v_4)$

<u>claim</u> $\{L_i \subset \Lambda^i V\}$ come from $Fl(V)$ lines

$\Leftrightarrow f_{ij}(L_i \otimes L_j) = 0$

$f_{11}(L_1 \otimes L_1) = L_1 \wedge L_1 = 0$

$\dim V = 4 \qquad v = \sum_{i<j} a_{ij} v_i \wedge v_j \in \Lambda^2 V$

$f_{22}(v, v) = a_{12} a_{34} - a_{13} a_{24} + a_{14} a_{23} = 0$

$Gr(2,4)$

$E_1 \subset \cdots \subset E_n = E \qquad G = GL_n$

$$Bun_B = \{E_1 \subset \cdots \subset E_n = E\}$$
$$= \{\underset{\text{sub-bundles}}{L_i \subset \Lambda^i E} \mid f_{ij}(L_i \otimes L_j) = 0\}$$

$$G/B \hookrightarrow \prod_{\omega_i} \mathbb{P}(V^{\omega_i})$$

ω_i fundamental weights

$$\lambda : T \to \mathbb{G}_m$$

$$\begin{tikzcd} & BT \arrow[d, "\lambda"] \\ C \arrow[ur, "P_T"] \arrow[r, "\mathcal{L}^\lambda_{P_T}"'] \arrow[ur, phantom, "\cong", shift right=2] & B\mathbb{G}_m \end{tikzcd}$$

$Bun_B(S)$

$$\Leftrightarrow \begin{cases} P_G \text{ on } C_S \\ P_T \text{ on } C_S \\ \kappa^\lambda : \mathcal{L}^\lambda_{P_T} \hookrightarrow V^\lambda_{P_G} \cong P_G \times_G V^\lambda \quad \text{for } \lambda \in \Lambda^+_G \text{ dominant} \end{cases}$$

s.t.

$$\begin{tikzcd} \mathcal{L}^{\lambda+\mu}_{P_T} \simeq \mathcal{L}^\lambda_{P_T} \otimes \mathcal{L}^\mu_{P_T} \arrow[r] \arrow[d] & V^\lambda_{P_G} \otimes V^\mu_{P_G} \arrow[dl, no head] \\ V^{\lambda+\mu}_{P_G} & \end{tikzcd}$$

$$\mathrm{Bun}_B \xrightarrow{p} \mathrm{Bun}_G$$
$$(P_G, P_T, \kappa) \longmapsto P_G$$
$$P_B \longmapsto P_B \times_B G$$

<u>Question</u> $p^!: D(\mathrm{Bun}_G) \to D(\mathrm{Bun}_B)$ is fully faithful?

<u>Answer</u> <u>No</u>

$$\begin{tikzcd}
\mathrm{Bun}_B \arrow[r] \arrow[dr, bend right, "p_{\mathrm{Bun}_B}"'] & \mathrm{Bun}_G \arrow[d, "p_{\mathrm{Bun}_G}"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & pt
\end{tikzcd}$$

$$\omega_{\mathcal{X}} = p_{\mathcal{X}}^! k \qquad p_{\mathcal{X}}: \mathcal{X} \to pt$$

$$\omega_{\mathrm{Bun}_G} \to \omega_{\mathrm{Bun}_B}$$

$$\underline{\mathrm{Hom}}(\omega_{\mathrm{Bun}_G}, \omega_{\mathrm{Bun}_G}) \overset{?}{\simeq} \mathrm{Hom}(\omega_{\mathrm{Bun}_B}, \omega_{\mathrm{Bun}_B})$$

<u>Take H^0</u>: $H^0_{dR}(\mathrm{Bun}_G) \overset{?}{\simeq} H^0_{dR}(\mathrm{Bun}_B)$

$B = T N$ $\Rightarrow$ $H^0(\mathrm{Bun}_B) = H^0(\mathrm{Bun}_T)$
contractible

$\pi_0(\mathrm{Bun}_G) = \pi_1(G)$
$\pi_0(\mathrm{Bun}_B) = \pi_1(T)$ } NOT SAME!

(e.g. for GL_2 $\mathbb{Z} \neq \mathbb{Z}^2$)

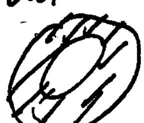
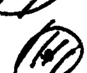
$B \hookrightarrow G$
$B \to T = B/N$

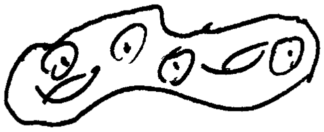
$\mathrm{Bun}_B \xrightarrow{q} \mathrm{Bun}_T$
$P_B \longmapsto P_B \times_B B/N$

<u>Goal:</u> Enhance p to find a fully-faithful embedding

<u>Idea</u>: (from number theory)
for $k = \mathbb{F}_q$
$\mathrm{Bun}_G(k) \simeq G(k(C)) \backslash G(\mathbb{A}) / G(\mathbb{O})$

In top'l context
$\mathrm{Bun}_G = L_{out}G \backslash LG / L_+G$

where $LG =$  $\to G$
$L_+G =$  $\to G$
$L_{out}G =$  $\to G$



$$G(\mathbb{A})/G(\mathbb{O})$$

$$\mathbb{A} = \prod^{res}_{x \in C} K_x \qquad K_x = \cancel{\mathcal{O}k}\ k((t))$$

$$\mathbb{O} = \prod_{x \in C} \mathcal{O}_x \qquad \mathcal{O}_x = k[[t]]$$

$$G(K_x)/G(\mathcal{O}_x) = Gr_G$$

$$\begin{tikzcd}
 & \boxed{?} \arrow[dl] & \\
Bun_G \arrow[d, equal] & & Bun_B \arrow[ll] \arrow[ul] \arrow[d, equal] \\
G(k)\backslash G(\mathbb{A})/G(\mathbb{O}) & & B(k)\backslash B(\mathbb{A})/B(\mathbb{O}) \arrow[ll] \arrow[dl] \\
 & B(k)\backslash G(\mathbb{A})/G(\mathbb{O}) \arrow[ul] &
\end{tikzcd}$$

Iwasawa decomposition

$$G(K_x) = B(K_x)\, G(\mathcal{O}_x)$$

$$k((t)) \to \begin{pmatrix} * & * \\ * & * \end{pmatrix} \qquad \uparrow\ k[[t]]$$

$$\rightsquigarrow G(\mathbb{A})/G(\mathbb{O}) = B(\mathbb{A})/B(\mathbb{O})$$

$H \subset G$ subgroup

$\mathrm{Bun}_G^{H\text{-gen}}$ is a prestack

$S \mapsto (P_G, U, \alpha_H)$

Where P_G is a principal G-bundle on C_S

$U \subset S_S$ is an open set

α_H is a reduction

$$\begin{tikzcd} & BH \arrow[d] \\ U \arrow[ur, dotted, "\alpha_H"] \arrow[r] & BG \end{tikzcd} \rightsquigarrow P_{H,U}$$

$(P_G, U, \alpha_H) \sim (P_G', U', \alpha_H')$

if on $U \cap U'$ they are isomorphic

$$\begin{tikzcd} P_H \arrow[r, mapsto] & (P_G, C, \alpha_H) & \\ & \mathrm{Bun}_G^{H\text{-gen}} \arrow[dr, bend left, "p_H^{enh}"] & \\ \mathrm{Bun}_H \arrow[ur, "i_H"] \arrow[rr] & & \mathrm{Bun}_G \\ P_H \arrow[rr, mapsto] & & P_G \end{tikzcd}$$

<u>Prop 1</u> $i_H : \mathrm{Bun}_H \to \mathrm{Bun}_G^{H\text{-gen}}$

induces an equivalence at the level of

k-pts, if H is parabolic

A parabolic subgroup P is anything in between $B \subseteq P \subseteq G$

<u>Pf</u> $U \to G/H$

can this extend to C?

G/H proper if H is parabolic

valuative criterion of properness $\boxtimes$

$$p_B^{enh} : Bun_G^{B\text{-gen}} \to Bun_G$$

$$Bun_G^{N\text{-gen}}$$

$$\underline{Map}(C, T)^{gen}$$

$$S \mapsto \left\{ \begin{matrix} U \xrightarrow{\gamma} T \\ \cap \\ S_S \end{matrix} \right.$$

generic maps

$$\begin{tikzcd} T \arrow[r] \arrow[d] & BN \arrow[d] \\ pt \arrow[r] & BB \end{tikzcd} \Rightarrow Bun_G^{N\text{-gen}} / \underline{Map}(C,T)^{gen} = Bun_G^{B\text{-gen}}$$

<u>Thm</u> (J. Barlev) $(p_B^{enh})^! : D(Bun_G) \to D(Bun_G^{B\text{-gen}})$

is fully faithful

Pf)
$$\begin{tikzcd} Bun_G^{1\text{-gen}} \times Map(C, G/B)^{gen} \arrow[r] \arrow[d] & Bun_G^{B\text{-gen}} \arrow[d, "p_{en}^{H}"] \\ Bun_G^{1\text{-gen}} \arrow[r] & Bun_G \end{tikzcd}$$

$$\begin{tikzcd} G(\mathbb{A})/G(\mathbb{O}) \times G(K)/B(K) \arrow[r] \arrow[d] & B(K)\backslash G(\mathbb{A})/G(\mathbb{O}) \arrow[d] \\ G(\mathbb{A})/G(\mathbb{O}) \arrow[r] & G(K)\backslash G(\mathbb{A})/G(\mathbb{O}) \end{tikzcd}$$

Claim/ $Map(C, G/B)^{gen}$ is trivial in some sense

Thm/ (Gaitsgory) If Y is nice (e.g. G/B, T)

$\Rightarrow$ $Map(C, Y)^{gen}$ is homologically contractible

$\Rightarrow$ ~~[illegible]~~

So now:
$$\begin{tikzcd} & Bun_G^{B\text{-gen}} \arrow[dr] & \\ Bun_B \arrow[ur] \arrow[rr] & & Bun_G \end{tikzcd}$$

for $G = GL_2$

$$\pi_0: \begin{tikzcd} & \mathbb{Z} \arrow[dr] & \\ \mathbb{Z}\times\mathbb{Z} \arrow[ur, "glued"] \arrow[rr] & & \mathbb{Z} \end{tikzcd}$$

how does this gluing happen?

$\underline{Ex}$ $G = GL_2$

$$Bun_B = \left\{ \underset{\text{sub-bundle}}{L \hookrightarrow E} \right\}$$

E/L bundle

$C = \mathbb{P}^1$

$$L = \mathcal{O} \underset{(x, x+t)}{\xrightarrow{\varphi_t}} E = \mathcal{O}(1) \oplus \mathcal{O}(1)$$

family of maps under t

For $t \neq 0$

$$0 \to \mathcal{O} \xrightarrow{h \mapsto (xh, (x+t)h)} \mathcal{O}(1) \oplus \mathcal{O}(1) \xrightarrow{f, g \mapsto (x+t)f - xg} \mathcal{O}(2) \to 0$$

for $t = 0$

$$0 \to \mathcal{O} \to \mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathcal{O}(1) \oplus k_0 \to 0$$

in affine coordinates

$$0 \to k[x] \xrightarrow{h \mapsto (xh, xh)} k[x] \oplus k[x] \xrightarrow{(f,g) \mapsto (f-g, g(0))} k[x] \oplus k \to 0$$

$k[x]/(x)$

$$\pi_0(Bun_B) = \pi_0(Bun_T) = \underset{\deg L}{\mathbb{Z}} \times \underset{\deg E/L}{\mathbb{Z}}$$

$t \neq 0 \Rightarrow (0, 2)$ $\Big\}$ differ by $(1, -1)$ ← coroot

$t = 0$ if one ignores $x = 0$, $(1, 1)$

"same component"

$\mathrm{Bun}_B \xrightarrow{p} \mathrm{Bun}_G$

p is not proper!

$\rightarrow \overline{\mathrm{Bun}_B}$ compactification

Drinfeld compactification

$(\mathcal{P}_G, \mathcal{P}_T, \kappa^\lambda)$

embedding of coh sheaves

then $\overline{\mathrm{Bun}_B} \xrightarrow{\bar{p}} \mathrm{Bun}_G$ is proper

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2) Global overview

$$\underbrace{\boxed{D(\mathrm{Bun}_G) \overset{?}{\simeq} IC_{N_{\check{G}}}(\mathrm{Flat}_{\check{G}})}}_{\text{The conjectured Geometric Langlands Correspondence}}$$

next week

hardest part of GLC!

$$\begin{tikzcd}
\boxed{\mathrm{Whit}^{ext.}(G)} \arrow[r, leftrightarrow] & \mathrm{Glue}(\check{G}) \\
D(\mathrm{Bun}_G) \arrow[u, hook, "!"] \arrow[r, phantom, "\overset{?}{\simeq}"] & IC_{N_{\check{G}}}(\mathrm{Flat}_{\check{G}}) \arrow[u, hook, "{[AG2]}"']
\end{tikzcd}$$

[AG1]

$\underline{\text{POINT}}$ The upper two categories are of local nature!

$\underbrace{\hspace{3cm}}$ to be explained in the final lecture on Geometric Satake

$$D(\mathrm{Bun}_G) \overset{?}{\simeq} IC(\mathrm{Flat}_{\check G})$$

KM-rep $\quad$ $\partial C(\mathrm{Op}_{\check G})$ opers

Conformal Field theory $\quad$ (via. Belinson-Drinfeld-Hitchin system)

Number theory: $M \leftarrow P \rightarrow G$

$G = (\ )$ $\quad P = (\ )$ $\quad M = (\ )$

$$\begin{array}{ccc} & \mathrm{Bun}_P & \\ {\scriptstyle q}\swarrow & & \searrow{\scriptstyle p} \\ \mathrm{Bun}_M & & \mathrm{Bun}_G \end{array} \qquad \begin{array}{ccc} & \mathrm{Flat}_{\check G} & \\ {\scriptstyle q^{Gal}}\swarrow & & \searrow{\scriptstyle p^{Gal}} \\ \mathrm{Flat}_{\check M} & & \mathrm{Flat}_{\check G} \end{array}$$

$\Rightarrow$ Can define geometric Eisenstein functor $\mathrm{Eis}_M(\mathcal{F}) = p_! q^* \mathcal{F}$ on D! $\Big|$ $\mathrm{Eis}_M^{Gal}(\mathcal{F}) = p_*^{Gal}(q^{Gal})^! \mathcal{F}$ on IC

$$\begin{tikzcd}
D(\mathrm{Bun}_G) \arrow[r, "\overset{?}{\simeq}", "\mathbb{L}_G"'] & IC_{N_{\check{G}}}(\mathrm{Flat}_{\check{G}}) \\
D(\mathrm{Bun}_M) \arrow[u, "\mathrm{Eis}_M"] \arrow[r, "\simeq", "\mathbb{L}_M"'] & IC_{N_{\check{M}}}(\mathrm{Flat}_{\check{M}}) \arrow[u, "\mathrm{Eis}_M^{Gal}"']
\end{tikzcd}$$

Kac-Moody rep.s + Eis generate D	opers + Eis^{Gal} generate $IC_{N_{\check{G}}}$

Everything here has interpretation under

$\mathcal{N} = 4$ SUSY Yang-Mills

$\boxed{10}$ Factorization Structures

<u>Goal</u>: Understan $D(\mathrm{Bun}_G)$

Recall $D(\mathrm{Bun}_G) \hookrightarrow D(\mathrm{Bun}_G^{B\text{-gen}})$

fully faithful

One can show $D(\mathrm{Bun}_G^{B\text{-gen}}) \hookrightarrow D(\mathrm{Bun}_G^{N\text{-gen}})$

<u>Question</u>: Why is $D(\mathrm{Bun}_G^{N\text{-gen}})$ easier given that $\mathrm{Bun}_G^{N\text{-gen}}$ is <u>NOT</u> Artin stack in general

BG is artin:

$$\cdots\ G\times G\times G \rightrightarrows G\times G \rightrightarrows G \to 1$$

"colimit of affine derived schemes w/ smooth morphisms"

<u>Answer</u>:

$$\begin{tikzcd} \exists\, \mathrm{Whit}(G) \arrow[r, hook] & D(\mathrm{Bun}_G^{N\text{-gen}}) \\ & D(\mathrm{Bun}_G) \arrow[u, hook] \arrow[ul, "!"] \arrow[ul, phantom, shift left=3, "\circlearrowright"] \end{tikzcd}$$

factorization structure

Whittaker coefficient